4 Identical Particles

In this chapter we consider the consequences resulting from the fact that identical particles are indistinguishable in quantum mechanics (as compared to classical mechanics), and the formalism needed to deal with such situations.

Some of the material presented in this chapter is taken from Auletta, Fortunato and Parisi, Chap. 7 and Cohen-Tannoudji, Diu and Laloë, Vol. II, Chaps. XIV.

4.1 Exchange Degeneracy

In classical mechanics even if two particles forming a system are identical (i.e., they share the same physical properties, such as mass, charge, etc.), there is nothing unusual to the manner with which one would go about analyzing the system's evolution. That is, the case of two identical particles is simply a special case of the more general situation when the particles are different. In other words, the fact that the particles are identical does not preclude us from following their respective evolutions separately. It is still possible to label the particles (say, particle 1 and particle 2) without any danger of confusion.

Things are very different in quantum mechanics since particles do not have definite trajectories, but rather they are described using wave functions. If two identical particles are separated by a distance that is much larger than the extent of their wave functions, such that they do not overlap, then it is presumably easy to label them and keep track of their evolution. But it is possible that their evolution would bring a spatial overlap of their wave functions (e.g., they could be in the process of colliding), and it then becomes impossible to tell them apart. More precisely, if we detect the presence of a particle in a region where both have a sizeable probability of being present (i.e., their wave functions are non-zero overlapping), it then becomes impossible to say which particle was detected. This situation is an example what is referred to ***exchange degeneracy***.

Exercise 4.1. Let us consider the case of two spin one-half particles. We assume that one particle is in the spin-up state $|+\rangle$, where its spin component along the z-axis has the $\hbar/2$ eigenvalue, while the other is in the spin-down state $|-\rangle$. Find the most general ket for this system and determine the probability of finding both particles having their spin component along the x-axis $\hat{S}_x$ with the $\hbar/2$ eigenvalue.

Solution.

The kets $|+,-\rangle$ and $|-,+\rangle$ both corresponds to a spin-up/spin-down state. The most general form of such state therefore consists of a linear combination of these kets of the form

$$|\psi\rangle = \alpha\,|+,-\rangle + \beta\,|-,+\rangle\,, \tag{4.1}$$

with $|\alpha|^2 + |\beta|^2 = 1$ to ensure normalization. We therefore find that there exists an infinite number of kets with the same physical state where one spin is up and the other down. This is clearly a case of exchange degeneracy.

We know from equations (3.127) and (3.129) of Chapter 3 that the matrix associated to the operator $\hat{S}_x$ is given by

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.2}$$

when using the basis

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.3}$$

We could formally diagonalize equation (4.2) and determine its eigenvectors using the material covered in Exercise 1.5 of Chapter 1, but it is straightforward to achieve this by inspection. It will be readily verified that the eigenvectors of $\hat{S}_x$ are given by

$$|\pm\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right), \tag{4.4}$$

i.e., $\hat{S}_x\,|\pm\rangle_x = \pm\hbar/2\,|\pm\rangle_x$. The compound state where both particles are in the $|+\rangle_x$ state is thus expressed by the following direct product

$$\begin{aligned} |\chi\rangle &= |+\rangle_x \otimes |+\rangle_x \\ &= \frac{1}{2}\left(|+,+\rangle + |+,-\rangle + |-,+\rangle + |-,-\rangle\right). \end{aligned} \tag{4.5}$$

The probability of finding the system in that state is

$$\begin{aligned} \mathcal{P}\left(+,+_x\right) &= \left|\langle\chi\,|\,\psi\rangle\right|^2 \\ &= \left|\frac{1}{2}\left(\alpha+\beta\right)\right|^2. \end{aligned} \tag{4.6}$$

This result is problematic because it implies that it is not possible to uniquely describe the state of the system. In other words, it implies that physicists performing independent experiments on similar systems would obtain results that are not consistent with each other. This is entirely non-sensical from a physics standpoint. This state of affair is entirely traceable to the exchange degeneracy, which must therefore be lifted in order for such a system to be describable in a way that is to be expected from a physical system. Evidently, the exchange degeneracy is not limited to systems composed of two identical particles, and can be generalized to any arbitrary number N of them.

4.2 Hamiltonian Invariance (or Symmetry)

If two identical particles are part of a system, then the corresponding Hamiltonian must be invariant (or symmetric) under the permutation of these particles. This could actually be used as a definition for identical particles. We label these two particles as 1 and 2 and define the ***permutation operator*** $\hat{P}_{21} = \hat{P}_{12}$ such that given a compound state $|\varphi(1)\,\chi(2)\rangle$ for the two particles we have

$$\begin{aligned} \hat{P}_{21}\,|\varphi(1)\,\chi(2)\rangle &= |\varphi(2)\,\chi(1)\rangle \\ &= |\chi(1)\,\varphi(2)\rangle\,. \end{aligned} \tag{4.7}$$

The fact that the permutation of two identical particles leaves the Hamiltonian (i.e., the energy) of a system unchanged implies that corresponding operators commute with each other

$$\left[\hat{P}_{21}, \hat{H}\right] = 0. \tag{4.8}$$

From this we conclude that $\hat{P}_{21}$ (or any permutation operator, for that matter) and $\hat{H}$ share the same set of eigenvectors (see Section 3.2 of Chapter 3). If we denote that basis as $\{|u_j(1)\,u_k(2)\rangle\}$, we can evaluate the matrix elements of $\hat{P}_{21}$ with

$$\begin{aligned} \left\langle u_j(1)\,u_k(2)\,\middle|\,\hat{P}_{21}\,\middle|\,u_m(1)\,u_n(2)\right\rangle &= \langle u_j(1)\,u_k(2)\,|\,u_n(1)\,u_m(2)\rangle \\ &= \delta_{jn}\delta_{km}. \end{aligned} \tag{4.9}$$

Likewise, the matrix elements of $\hat{P}_{21}^{\dagger}$ are calculated to be

$$\begin{aligned} \left\langle u_j(1)\,u_k(2)\,\middle|\,\hat{P}_{21}^{\dagger}\,\middle|\,u_m(1)\,u_n(2)\right\rangle &= \langle u_k(1)\,u_j(2)\,|\,u_m(1)\,u_n(2)\rangle \\ &= \delta_{km}\delta_{jn} \end{aligned} \tag{4.10}$$

since $\left(\hat{P}_{21}\,|u_j(1)\,u_k(2)\rangle\right)^{\dagger} = \langle u_j(1)\,u_k(2)|\,\hat{P}_{21}^{\dagger}$. A comparison of equations (4.9) and (4.10) reveals that this permutation operator is Hermitian, i.e., $\hat{P}_{21}^{\dagger} = \hat{P}_{21}$. Also, since it is clear that

$$\left(\hat{P}_{21}\right)^2 = \hat{1}, \tag{4.11}$$

we find that $\hat{P}_{21}^{\dagger}\hat{P}_{21} = \hat{1}$. The permutation operator is therefore unitary.

Exercise 4.2. Use the basis of eigenvectors $\{|u_j(1)\,u_k(2)\rangle\}$ common to $\hat{H}$ and $\hat{P}_{21}$ to find an expression for the latter.

Solution.

Let us use the closure relation $\sum_{j,k} |u_j(1)\, u_k(2)\rangle\langle u_j(1)\, u_k(2)| = \hat{1}$ to expand the following ($|\psi\rangle$ is an arbitrary state vector)

$$\begin{aligned}
\hat{P}_{21}|\psi\rangle &= \hat{P}_{21}\sum_{j,k}|u_j(1)\, u_k(2)\rangle\, \langle u_j(1)\, u_k(2)\,|\,\psi\rangle \\
&= \sum_{j,k}\hat{P}_{21}|u_j(1)\, u_k(2)\rangle\, \langle u_j(1)\, u_k(2)\,|\,\psi\rangle \\
&= \sum_{j,k}|u_k(1)\, u_j(2)\rangle\, \langle u_j(1)\, u_k(2)\,|\,\psi\rangle\,, \qquad (4.12)
\end{aligned}$$

which implies that

$$\begin{aligned}
\hat{P}_{21} &= \sum_{j,k}|u_k(1)\,, u_j(2)\rangle\langle u_j(1)\,, u_k(2)| \\
&= \sum_{j,k}|u_j(1)\,, u_k(2)\rangle\langle u_k(1)\,, u_j(2)|\,. \qquad (4.13)
\end{aligned}$$

Although we have so far limited our discussion to a finite basis, the same reasoning applies to infinite, continuous bases or a combination of both. For example, using the compound basis $\{|\mathbf{r}_1,\varepsilon_1;\mathbf{r}_2,\varepsilon_2\rangle\}$ for the positions $\hat{\mathbf{r}}_j$ and spin components $\hat{S}_{jz}$ of two identical particles ($j = 1, 2$), it is also possible to determine the effect of $\hat{P}_{21}$ on their wave function $\psi(\mathbf{r}_1,\varepsilon_1;\mathbf{r}_2,\varepsilon_2)$. That is, if we define

$$|\varphi\rangle = \hat{P}_{21}|\psi\rangle \qquad (4.14)$$

we can calculate (since $\hat{P}_{21} = \hat{P}_{21}^{\dagger}$)

$$\begin{aligned}
\varphi(\mathbf{r}_1,\varepsilon_1;\mathbf{r}_2,\varepsilon_2) &= \left\langle \mathbf{r}_1,\varepsilon_1;\mathbf{r}_2,\varepsilon_2 \middle| \hat{P}_{21} \middle| \psi \right\rangle \\
&= \langle \mathbf{r}_2,\varepsilon_2;\mathbf{r}_1,\varepsilon_1\,|\,\psi\rangle \\
&= \psi(\mathbf{r}_2,\varepsilon_2;\mathbf{r}_1,\varepsilon_1) \qquad (4.15)
\end{aligned}$$

And in a similar manner as with a finite basis only, we find that

$$\hat{P}_{21} = \sum_{\varepsilon_1,\varepsilon_2}\int_{-\infty}^{\infty} d^3x d^3x'\,|\mathbf{r}_1,\varepsilon_1;\mathbf{r}_2,\varepsilon_2\rangle\langle\mathbf{r}_2,\varepsilon_2;\mathbf{r}_1,\varepsilon_1|\,. \qquad (4.16)$$

4.2.1 Symmetric and Antisymmetric States

We already know from equation (4.11) that, for $|\psi\rangle$ one of its eigenvector,

$$\left(\hat{P}_{21}\right)^2|\psi\rangle = |\psi\rangle\,, \qquad (4.17)$$

and since $\hat{P}_{21}$ is Hermitian and must therefore have real eigenvalues, it follows that

$$\hat{P}_{21} |\psi\rangle = \pm |\psi\rangle \,. \tag{4.18}$$

That is, ± 1 are the only possible eigenvalues for $\hat{P}_{21}$. The eigenvectors $\{|\psi_S\rangle, |\psi_A\rangle\}$ such that

$$\begin{aligned} \hat{P}_{21} |\psi_S\rangle &= +|\psi_S\rangle & (4.19) \\ \hat{P}_{21} |\psi_A\rangle &= -|\psi_A\rangle & (4.20) \end{aligned}$$

are respectively called ***symmetric*** and ***antisymmetric states*** since they are symmetric and antisymmetric to the permutation of two identical particles (i.e., the kets change sign or not after the permutation).

We can define the projectors on the spaces to which the symmetric and antisymmetric states belong with

$$\begin{aligned} \hat{S} &= \frac{1}{2}\left(\hat{1} + \hat{P}_{21}\right) & (4.21) \\ \hat{A} &= \frac{1}{2}\left(\hat{1} - \hat{P}_{21}\right), & (4.22) \end{aligned}$$

since for an arbitrary ket (as always $|\alpha|^2 + |\beta|^2 = 1$)

$$|\psi\rangle = \alpha |\psi_S\rangle + \beta |\psi_A\rangle \tag{4.23}$$

we have

$$\begin{aligned} \hat{S} |\psi\rangle &= \frac{1}{2}\left(\hat{1} + \hat{P}_{21}\right)(\alpha |\psi_S\rangle + \beta |\psi_A\rangle) \\ &= \alpha |\psi_S\rangle & (4.24) \\ \hat{A} |\psi\rangle &= \frac{1}{2}\left(\hat{1} - \hat{P}_{21}\right)(\alpha |\psi_S\rangle + \beta |\psi_A\rangle) \\ &= \beta |\psi_A\rangle \,. & (4.25) \end{aligned}$$

It is also straightforward to verify that

$$\begin{aligned} \hat{S}^\dagger &= \hat{S} & (4.26) \\ \hat{A}^\dagger &= \hat{A} & (4.27) \\ \hat{S}^2 &= \hat{S} & (4.28) \\ \hat{A}^2 &= \hat{A}, & (4.29) \end{aligned}$$

and

$$\begin{aligned}
\left[\hat{S}, \hat{A}\right] &= \hat{0} & (4.30)\\
\left[\hat{S}, \hat{P}_{21}\right] &= \hat{0} & (4.31)\\
\left[\hat{A}, \hat{P}_{21}\right] &= \hat{0} & (4.32)\\
\hat{S} + \hat{A} &= \hat{1}. & (4.33)
\end{aligned}$$

The combinations of equations (4.24) with (4.31) and (4.25) with (4.32), respectively, yield

$$\begin{aligned}
\hat{P}_{21}\hat{S}\,|\psi\rangle &= \hat{S}\hat{P}_{21}\,|\psi\rangle \\
&= \hat{S}\,|\psi\rangle & (4.34)\\
\hat{P}_{21}\hat{A}\,|\psi\rangle &= \hat{A}\hat{P}_{21}\,|\psi\rangle \\
&= -\hat{A}\,|\psi\rangle & (4.35)
\end{aligned}$$

We thus find that $\hat{S}$ and $\hat{A}$ are ***symmetrizer*** and ***antisymmetrizer operators***, respectively.

Exercise 4.3. We consider a system containing two identical particles 1 and 2. Show that for the extended observables $\hat{\mathbb{B}}(j)$ and $\hat{\mathbb{C}}(j)$, with $j = 1, 2$, we have

$$\begin{aligned}
\hat{P}_{21}\hat{\mathbb{B}}(1)\,\hat{P}_{21}^{\dagger} &= \hat{\mathbb{B}}(2) & (4.36)\\
\hat{P}_{21}\left[\hat{\mathbb{B}}(1) + \hat{\mathbb{C}}(2)\right]\hat{P}_{21}^{\dagger} &= \hat{\mathbb{B}}(2) + \hat{\mathbb{C}}(1) & (4.37)\\
\hat{P}_{21}\hat{\mathbb{B}}(1)\,\hat{\mathbb{C}}(2)\,\hat{P}_{21}^{\dagger} &= \hat{\mathbb{B}}(2)\,\hat{\mathbb{C}}(1). & (4.38)
\end{aligned}$$

Solution.

To simplify the calculations we will use a basis $\{|u_j(1)\,u_k(2)\rangle\}$ containing the eigenvectors of the extended observables $\hat{\mathbb{B}}(1)$ and $\hat{\mathbb{B}}(2)$ (the corresponding eigenvalues are b_j and b_k, respectively). We then have

$$\begin{aligned}
\hat{P}_{21}\hat{\mathbb{B}}(1)\,\hat{P}_{21}^{\dagger}\,|u_j(1)\,u_k(2)\rangle &= \hat{P}_{21}\hat{\mathbb{B}}(1)\,|u_k(1)\,u_j(2)\rangle \\
&= b_k\hat{P}_{21}\,|u_k(1)\,u_j(2)\rangle \\
&= b_k\,|u_j(1)\,u_k(2)\rangle \\
&= \hat{\mathbb{B}}(2)\,|u_j(1)\,u_k(2)\rangle, & (4.39)
\end{aligned}$$

and $\hat{P}_{21}\hat{\mathbb{B}}(1)\,\hat{P}_{21}^{\dagger} = \hat{\mathbb{B}}(2)$. Since the same technique can be applied to the observables $\hat{\mathbb{C}}(1)$ and $\hat{\mathbb{C}}(2)$, it follows that $\hat{P}_{21}\left[\hat{\mathbb{B}}(1) + \hat{\mathbb{C}}(2)\right]\hat{P}_{21}^{\dagger} = \hat{\mathbb{B}}(2) + \hat{\mathbb{C}}(1)$. For equation (4.38) we can write

$$\begin{aligned}\hat{P}_{21}\hat{\mathbb{B}}(1)\,\hat{\mathbb{C}}(2)\,\hat{P}_{21}^{\dagger} &= \hat{P}_{21}\hat{\mathbb{B}}(1)\,\hat{P}_{21}^{\dagger}\hat{P}_{21}\hat{\mathbb{C}}(2)\,\hat{P}_{21}^{\dagger}\\ &= \hat{\mathbb{B}}(2)\,\hat{\mathbb{C}}(1)\,. \end{aligned}\tag{4.40}$$

Finally, we note that the last two relations can be generalized to any operator $\hat{\mathbb{O}}(1,2)$ such that

$$\hat{P}_{21}\hat{\mathbb{O}}(1,2)\,\hat{P}_{21}^{\dagger} = \hat{\mathbb{O}}(2,1)\,. \tag{4.41}$$

An observable is said to be symmetric when $\hat{\mathbb{O}}_S(1,2) = \hat{\mathbb{O}}_S(2,1)$, and from equation (4.41) we find that

$$\left[\hat{\mathbb{O}}_S(1,2)\,,\hat{P}_{21}\right] = 0. \tag{4.42}$$

4.2.2 Generalization to an Arbitrary Number of Particles

We have so far limited ourselves to the special case of two identical particles. The process can be extended to permutations between an arbitrary number of N particles but, as we will see, we should be careful that not all of the properties obtained for the case of two identical particles apply in general. It will be easier if we first look at the $N = 3$ case before generalizing our results.

For a system composed of three identical particles (e.g., three electrons or the H_3^+ molecule) a state for the system can be written as

$$|\psi\rangle = |u_i(1)\,u_j(2)\,u_k(3)\rangle\,, \tag{4.43}$$

where $\{u_i\}$ is some basis valid for all particles (since they are identical). The permutation operator $\hat{P}_{mnp}$ is such that it transforms the state vector by replacing the particle with label m with label n, the one with label n is replaced by label p and the one with label p by label m.[1] For example, we have

$$\begin{aligned}\hat{P}_{132}\,|u_i(1)\,u_j(2)\,u_k(3)\rangle &= |u_i(3)\,u_j(1)\,u_k(2)\rangle\\ &= |u_j(1)\,u_k(2)\,u_i(3)\rangle\,. \end{aligned}\tag{4.44}$$

There exists $N!$ permutations, i.e., six for $N = 3$, $\hat{P}_{123}$, $\hat{P}_{132}$, $\hat{P}_{12}$, $\hat{P}_{23}$, $\hat{P}_{31}$ and the identity $\hat{1}$. We should note that performing a cyclic permutation on a given operator leaves it unchanged, i.e., $\hat{P}_{123} = \hat{P}_{231} = \hat{P}_{312}$, etc. Also, every permutation admits an inverse operation and the product of two permutations yields another permutation, e.g.,

[1] There are different ways of defining the permutation operator when $N > 2$. I follow here the one used by, e.g., Auletta, Fortunato and Parisi.

$$\begin{aligned} \hat{P}_{213}\hat{P}_{123}\left|u_i(1)\,u_j(2)\,u_k(3)\right\rangle &= \hat{P}_{213}\left|u_i(2)\,u_j(3)\,u_k(1)\right\rangle \\ &= \left|u_i(1)\,u_j(2)\,u_k(3)\right\rangle \end{aligned} \tag{4.45}$$

such that $\hat{P}_{213} = \hat{P}_{123}^{-1}$, and

$$\begin{aligned} \hat{P}_{12}\hat{P}_{123}\left|u_i(1)\,u_j(2)\,u_k(3)\right\rangle &= \hat{P}_{12}\left|u_i(2)\,u_j(3)\,u_k(1)\right\rangle \\ &= \left|u_i(1)\,u_j(3)\,u_k(2)\right\rangle \end{aligned} \tag{4.46}$$

or $\hat{P}_{12}\hat{P}_{123} = \hat{P}_{23}$. Evidently, a permutation of two particles only, which is called a ***transposition operator***, is its own inverse (i.e., $\hat{P}_{12} = \hat{P}_{12}^{-1}$). Generally, for an arbitrary number of particles N the inverse of a permutation operator is given by

$$\hat{P}_{abc\cdots yz}^{-1} = \hat{P}_{azy\cdots cb}. \tag{4.47}$$

Moreover, a permutation can always be broken down into a product of transpositions in the following manner

$$\hat{P}_{abcde} = \hat{P}_{ab}\hat{P}_{bc}\hat{P}_{cd}\hat{P}_{de}. \tag{4.48}$$

A given permutation is called ***even*** or ***odd*** depending on whether it has an even or odd number of transpositions in its transposition product (the permutation of equation (4.48) is even); this defines the ***parity*** of the permutation. It follows that ***permutation operators are unitary***, since they are the product of transposition operators (which are unitary).

In general the product of two permutations does not depend on what vector it is applied to, and it is possible to evaluate it without having recourse to an ket. For example, it can be said that on the left hand side of equation (4.46): 1 is first replaced by 2 (the $\hat{P}_{123}$ permutation) and then by 1 (the $\hat{P}_{12}$ permutation); 2 is first replaced by 3 (the $\hat{P}_{123}$ permutation) and then by 3 (the $\hat{P}_{12}$ permutation); 3 is first replaced by 1 (the $\hat{P}_{123}$ permutation) and then by 2 (the $\hat{P}_{12}$ permutation). Summarizing all this we have

$$1 \leftarrow 2 \leftarrow 1 = 1 \leftarrow 1, \qquad 2 \leftarrow 3 \leftarrow 3 = 2 \leftarrow 3, \qquad 3 \leftarrow 1 \leftarrow 2 = 3 \leftarrow 2, \tag{4.49}$$

or $\hat{P}_{12}\hat{P}_{123} = \hat{P}_{23}$, as expected. Using the same procedure we find that $\hat{P}_{123}\hat{P}_{12} = \hat{P}_{13}$ and, therefore, $\hat{P}_{12}\hat{P}_{123} \neq \hat{P}_{123}\hat{P}_{12}$. This result is true in general, i.e., ***permutation operators do not commute*** (i.e., when $N > 2$). In particular, we note that $\hat{P}_{12}\hat{P}_{23} \neq \hat{P}_{23}\hat{P}_{12}$, which should be apparent from $\hat{P}_{12}\hat{P}_{23} = \hat{P}_{123}$ and $\hat{P}_{23}\hat{P}_{12} = \hat{P}_{32}\hat{P}_{21} = \hat{P}_{321}$ and therefore

$$\begin{aligned} \hat{P}_{123}^{\dagger} &= \left(\hat{P}_{12}\hat{P}_{23}\right)^{\dagger} \\ &= \hat{P}_{23}^{\dagger}\hat{P}_{12}^{\dagger} \\ &= \hat{P}_{23}\hat{P}_{12} \\ &= \hat{P}_{321}. \end{aligned} \tag{4.50}$$

We thus conclude that ***permutation operator are not Hermitian in general*** (i.e., when $N > 2$).

To determine the effect of a permutation operator on a wave function, it is advantageous to generalize the notation used so far. We start by writing $|\mathbf{r}_1, \varepsilon_1; \mathbf{r}_2, \varepsilon_2; \mathbf{r}_3, \varepsilon_3\rangle = |\mathbf{r}_1, \varepsilon_1 (1) \, \mathbf{r}_2, \varepsilon_2 (2) \, \mathbf{r}_3, \varepsilon_3 (3)\rangle$ (we choose $N = 3$, for simplicity) and with, say, $|\varphi\rangle = \hat{P}_{123} |\psi\rangle$,

$$\begin{aligned}
\varphi (\mathbf{r}_1, \varepsilon_1; \mathbf{r}_2, \varepsilon_2; \mathbf{r}_3, \varepsilon_3) &= \left\langle \mathbf{r}_1, \varepsilon_1 (1) \, \mathbf{r}_2, \varepsilon_2 (2) \, \mathbf{r}_3, \varepsilon_3 (3) \left| \hat{P}_{123} \right| \psi \right\rangle \\
&= \left[\hat{P}_{123}^{\dagger} |\mathbf{r}_1, \varepsilon_1 (1) \, \mathbf{r}_2, \varepsilon_2 (2) \, \mathbf{r}_3, \varepsilon_3 (3)\rangle \right]^{\dagger} |\psi\rangle \\
&= \left[\hat{P}_{132} |\mathbf{r}_1, \varepsilon_1 (1) \, \mathbf{r}_2, \varepsilon_2 (2) \, \mathbf{r}_3, \varepsilon_3 (3)\rangle \right]^{\dagger} |\psi\rangle \\
&= \left[|\mathbf{r}_1, \varepsilon_1 (3) \, \mathbf{r}_2, \varepsilon_2 (1) \, \mathbf{r}_3, \varepsilon_3 (2)\rangle \right]^{\dagger} |\psi\rangle \\
&= \langle \mathbf{r}_2, \varepsilon_2 (1) \, \mathbf{r}_3, \varepsilon_3 (2) \, \mathbf{r}_1, \varepsilon_1 (3) \, | \, \psi\rangle \\
&= \psi (\mathbf{r}_2, \varepsilon_2; \mathbf{r}_3, \varepsilon_3; \mathbf{r}_1, \varepsilon_1)
\end{aligned} \tag{4.51}$$

For example, if we set

$$\psi (\mathbf{r}_1, \varepsilon_1; \mathbf{r}_2, \varepsilon_2; \mathbf{r}_3, \varepsilon_3) = r_1 + 2r_2 + 3r_3 \tag{4.52}$$

then

$$\begin{aligned}
\varphi (\mathbf{r}_1, \varepsilon_1; \mathbf{r}_2, \varepsilon_2; \mathbf{r}_3, \varepsilon_3) &= \psi (\mathbf{r}_2, \varepsilon_2; \mathbf{r}_3, \varepsilon_3; \mathbf{r}_1, \varepsilon_1) \\
&= r_2 + 2r_3 + 3r_1.
\end{aligned} \tag{4.53}$$

We should be careful to note that in this case we applied the permutation operator $\hat{P}_{123}$ to the ket $|\psi\rangle$, not the eigenvector $|\mathbf{r}_1, \varepsilon_1 (1) \, \mathbf{r}_2, \varepsilon_2 (2) \, \mathbf{r}_3, \varepsilon_3 (3)\rangle$. The two situations are not equivalent. More precisely, if we apply the permutation operator $\hat{P}_{123}$ to the eigenvector $|\mathbf{r}_1, \varepsilon_1 (1) \, \mathbf{r}_2, \varepsilon_2 (2) \, \mathbf{r}_3, \varepsilon_3 (3)\rangle$ we instead have

$$\begin{aligned}
\varphi' (\mathbf{r}_1, \varepsilon_1; \mathbf{r}_2, \varepsilon_2; \mathbf{r}_3, \varepsilon_3) &= \left[\hat{P}_{123} |\mathbf{r}_1, \varepsilon_1 (1) \, \mathbf{r}_2, \varepsilon_2 (2) \, \mathbf{r}_3, \varepsilon_3 (3)\rangle \right]^{\dagger} |\psi\rangle \\
&= \left\langle \mathbf{r}_1, \varepsilon_1 (1) \, \mathbf{r}_2, \varepsilon_2 (2) \, \mathbf{r}_3, \varepsilon_3 (3) \left| \hat{P}_{123}^{\dagger} \right| \psi \right\rangle \\
&= \left\langle \mathbf{r}_1, \varepsilon_1 (1) \, \mathbf{r}_2, \varepsilon_2 (2) \, \mathbf{r}_3, \varepsilon_3 (3) \left| \hat{P}_{132} \right| \psi \right\rangle \\
&= \langle \mathbf{r}_3, \varepsilon_3 (1) \, \mathbf{r}_1, \varepsilon_1 (2) \, \mathbf{r}_2, \varepsilon_2 (3) \, | \, \psi\rangle \\
&= \psi (\mathbf{r}_3, \varepsilon_3; \mathbf{r}_1, \varepsilon_1; \mathbf{r}_2, \varepsilon_2) \, ,
\end{aligned} \tag{4.54}$$

and with the previous example

$$\begin{aligned}
\varphi (\mathbf{r}_1, \varepsilon_1; \mathbf{r}_2, \varepsilon_2; \mathbf{r}_3, \varepsilon_3) &= \psi (\mathbf{r}_3, \varepsilon_3; \mathbf{r}_1, \varepsilon_1; \mathbf{r}_2, \varepsilon_2) \\
&= r_3 + 2r_1 + 3r_2.
\end{aligned} \tag{4.55}$$

Evidently, we have $\varphi'(\mathbf{r}_1,\varepsilon_1;\mathbf{r}_2,\varepsilon_2;\mathbf{r}_3,\varepsilon_3) \neq \varphi(\mathbf{r}_1,\varepsilon_1;\mathbf{r}_2,\varepsilon_2;\mathbf{r}_3,\varepsilon_3)$. In fact, as is apparent from our calculations, these two wave functions can be seen to result from the action on the ket $|\psi\rangle$ of permutation operators that are inverse from one another (i.e., $\hat{P}_{132}$ and $\hat{P}_{123}$). This is akin to either effecting a rotation on a system of coordinate axes or on an object (e.g., a vector) positioned in relation to these axes.

For N arbitrary, we define a ***totally symmetric*** ket as one for that is left unchanged by all permutation operator, i.e.,

$$\hat{P}_\alpha |\psi_S\rangle = |\psi_S\rangle, \tag{4.56}$$

where α represents some permutation. Conversely, a ***totally antisymmetric*** ket is defined by

$$\hat{P}_\alpha |\psi_A\rangle = \epsilon_\alpha |\psi_A\rangle, \tag{4.57}$$

where

$$\epsilon_\alpha = \begin{cases} +1, & \text{if } \hat{P}_\alpha \text{ is of even parity} \\ -1, & \text{if } \hat{P}_\alpha \text{ is of odd parity.} \end{cases} \tag{4.58}$$

For N identical particles, $N/2$ have $\epsilon_\alpha = \pm 1$.

We can now generalize the ***symmetrizer*** and ***antisymmetrizer*** operators previously introduced for $N = 2$ with

$$\hat{S} = \frac{1}{N!}\sum_\alpha \hat{P}_\alpha \tag{4.59}$$

$$\hat{A} = \frac{1}{N!}\sum_\alpha \epsilon_\alpha \hat{P}_\alpha. \tag{4.60}$$

Because permutation operators can be decomposed into a product of transpositions, $\hat{P}_\alpha^\dagger$ has the same number of transpositions as $\hat{P}_\alpha$ (and therefore the same parity), and that the summations in equations (4.59) and (4.60) are on all permutation operators, we find that $\hat{S}^\dagger = \hat{S}$ and $\hat{A}^\dagger = \hat{A}$. Moreover, since for any two permutations operators $\hat{P}_\alpha$ and $\hat{P}_\beta$ we have

$$\hat{P}_\beta \hat{P}_\alpha = \hat{P}_\gamma \tag{4.61}$$

$$\epsilon_\beta \epsilon_\alpha = \epsilon_\gamma, \tag{4.62}$$

it follows that

$$\begin{aligned} \hat{P}_\beta \hat{S} &= \hat{S}\hat{P}_\beta \\ &= \hat{S} \end{aligned} \tag{4.63}$$

$$\begin{aligned} \hat{P}_\beta \hat{A} &= \hat{A}\hat{P}_\beta \\ &= \epsilon_\beta \hat{A}, \end{aligned} \tag{4.64}$$

and

$$
\begin{aligned}
\hat{S}^2 &= \frac{1}{N!}\sum_\alpha \hat{P}_\alpha \hat{S} \\
&= \frac{1}{N!}\sum_\alpha \hat{S} \\
&= \hat{S} && (4.65) \\
\hat{A}^2 &= \frac{1}{N!}\sum_\alpha \epsilon_\alpha \hat{P}_\alpha \hat{A} \\
&= \frac{1}{N!}\sum_\alpha \epsilon_\alpha^2 \hat{A} \\
&= \hat{A} && (4.66) \\
\hat{A}\hat{S} &= \frac{1}{N!}\sum_\alpha \epsilon_\alpha \hat{P}_\alpha \hat{S} \\
&= \hat{S}\left(\frac{1}{N!}\sum_\alpha \epsilon_\alpha\right) \\
&= 0, && (4.67)
\end{aligned}
$$

since one half of the permutation operators have $\epsilon_\alpha = 1$ and the other half $\epsilon_\alpha = -1$. These last three relations imply that $\hat{S}$ and $\hat{A}$ are projectors. That is, from equations (4.63) and (4.64)

$$
\begin{aligned}
\hat{P}_\beta \hat{S} |\psi\rangle &= \hat{S} |\psi\rangle && (4.68) \\
\hat{P}_\beta \hat{A} |\psi\rangle &= \epsilon_\beta \hat{A} |\psi\rangle , && (4.69)
\end{aligned}
$$

and from equations (4.56) and (4.57)

$$
\begin{aligned}
\hat{S} |\psi\rangle &= |\psi_S\rangle && (4.70) \\
\hat{A} |\psi\rangle &= |\psi_A\rangle . && (4.71)
\end{aligned}
$$

Exercise 4.4. Consider the case where $N = 3$, and calculate $\hat{S}$, $\hat{A}$ and $\hat{S} + \hat{A}$.

Solution.

Since $\hat{1}$, $\hat{P}_{123}$ and $\hat{P}_{132}$ have $\epsilon = 1$ (0, 2 and 2 transpositions), while $\hat{P}_{12}$, $\hat{P}_{23}$ and $\hat{P}_{13}$ have $\epsilon = -1$ (1 transposition), we have

$$\begin{aligned}
\hat{S} &= \frac{1}{6}\left(\hat{1} + \hat{P}_{123} + \hat{P}_{132} + \hat{P}_{12} + \hat{P}_{23} + \hat{P}_{13}\right) && (4.72)\\
\hat{A} &= \frac{1}{6}\left(\hat{1} + \hat{P}_{123} + \hat{P}_{132} - \hat{P}_{12} - \hat{P}_{23} - \hat{P}_{13}\right) && (4.73)\\
\hat{S} + \hat{A} &= \frac{1}{3}\left(\hat{1} + \hat{P}_{123} + \hat{P}_{132}\right)\\
&\neq \hat{1}. && (4.74)
\end{aligned}$$

The result obtained in equation (4.74) for $N = 3$ is different than what we got when $N = 2$ (see equation (4.33)), and it implies that not all kets of the basis $\{|u_i(1)\,u_j(2)\,u_k(3)\rangle\}$ are covered by the $\hat{S}$ and $\hat{A}$ projectors.

4.3 Eight Postulate

It is found through experiments that when a system is composed of several identical particles not all state vectors contained in the basis resulting from the direct products of the individual particles' bases are realized physically (i.e., $\{|u_i(1)\,u_j(2)\,u_k(3)\rangle\}$ for $N = 3$), but only vectors that are either totally symmetric or antisymmetric. This fact leads to the ***Postulate of Symmetrization***

The state of a quantum mechanical system composed of N identical particles are either completely symmetric or completely antisymmetric with respect to the permutations of these particles. ***Bosons*** *are the particles for which the states are symmetric, while* ***fermions*** *are those for which the states are antisymmetric.*

It is further observed that ***boson have integer spin*** and ***fermion have half-integer spin***.

It is important to realize that restricting the states of a system to those of the symmetric and antisymmetric spaces associated to $\hat{S}$ and $\hat{A}$, respectively, lift the exchange degeneracy. This is because the degeneracy arises from the fact that all vectors link through permutations (e.g., $|u_i(1)\,u_j(2)\,u_k(3)\rangle$, $\hat{P}_{123}\,|u_i(1)\,u_j(2)\,u_k(3)\rangle$, etc.), which lead to the same physical state, will all be contained in a well defined way in these spaces. For example, we know from equations (4.68) and (4.69) that two kets $|\psi\rangle$ and $\hat{P}_\beta\,|\psi\rangle$ yield collinear states when acted upon by $\hat{S}$ or $\hat{A}$.

Exercise 4.5. Let us return to the case of two spin-1/2 particles, one particle in the spin-up state $|+\rangle$ and the other in the spin-down state $|-\rangle$. Determine the proper state for this system and calculate the probability of finding both particles having their spin component along the x-axis $\hat{S}_x$ with the $\hbar/2$ eigenvalue.

Solution.

Since we are dealing with fermions, we must build an antisymmetric state for this system. For this we use

$$\hat{A} = \frac{1}{2}\left(\hat{1} - \hat{P}_{21}\right), \tag{4.75}$$

and we can calculate an antisymmetric vector with

$$\begin{aligned} \left|\psi'_A\right\rangle &= \hat{A}\left|+,-\right\rangle \\ &= \frac{1}{2}\left(\hat{1} - \hat{P}_{21}\right)\left|+,-\right\rangle \\ &= \frac{1}{2}\left(\left|+,-\right\rangle - \left|-,+\right\rangle\right). \end{aligned} \tag{4.76}$$

It is clear that this ket is antisymmetric in the exchange of the two particles. We should normalize that ket to obtain

$$\left|\psi_A\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+,-\right\rangle - \left|-,+\right\rangle\right). \tag{4.77}$$

Projecting this ket on that from equation (4.5), which we write down for convenience here

$$\left|\chi\right\rangle = \frac{1}{2}\left(\left|+,+\right\rangle + \left|+,-\right\rangle + \left|-,+\right\rangle + \left|-,-\right\rangle\right), \tag{4.78}$$

we get

$$\begin{aligned} \mathcal{P}\left(+,+_x\right) &= \left|\left\langle \chi \,|\, \psi_A\right\rangle\right|^2 \\ &= 0. \end{aligned} \tag{4.79}$$

This result is now a clear prediction that does not exhibit any of the problems we originally saw in Exercise 4.1. The exchange degeneracy is lifted.

Incidentally, the result obtained in Exercise 4.5 showing a zero probability of finding the two fermions with the same spin component along the x-axis $\hat{S}_x$ is a manifestation of the ***Pauli Exclusion Principle***

Two identical fermions cannot occupy in the same individual state.

We can further test this principle on the $\left|+,+\right\rangle$ symmetric state. That is, we can verify that

$$\begin{aligned} \hat{A}\left|+,+\right\rangle &= \frac{1}{2}\left(\hat{1} - \hat{P}_{21}\right)\left|+,+\right\rangle \\ &= \frac{1}{2}\left(\left|+,+\right\rangle - \left|+,+\right\rangle\right) \\ &= 0, \end{aligned} \tag{4.80}$$

and, again, two identical fermions cannot both be in the $\left|+\right\rangle$ state.

We now consider three identical fermions ($N = 3$), which could potentially occupy three different states $|u_1\rangle$, $|u_2\rangle$ and $|u_3\rangle$. An appropriate antisymmetric state can be obtained with

$$\hat{A}\,|u_1\,(1)\,u_2\,(2)\,u_3\,(3)\rangle = \frac{1}{3!}\sum_{\alpha}\epsilon_{\alpha}\hat{P}_{\alpha}\,|u_1\,(1)\,u_2\,(2)\,u_3\,(3)\rangle\,. \tag{4.81}$$

It turns out that the alternations in the sign ϵ_α of the different terms are determined in the same manner as those for the determinant of a 3×3 matrix. It is therefore advantageous to write the antisymmetric ket in the form

$$\hat{A}\,|u_1\,(1)\,u_2\,(2)\,u_3\,(3)\rangle = \frac{1}{3!}\begin{vmatrix} |u_1\,(1)\rangle & |u_2\,(1)\rangle & |u_3\,(1)\rangle \\ |u_1\,(2)\rangle & |u_2\,(2)\rangle & |u_3\,(2)\rangle \\ |u_1\,(3)\rangle & |u_2\,(3)\rangle & |u_3\,(3)\rangle \end{vmatrix}. \tag{4.82}$$

For an arbitrary number of identical particles, while applying adequate normalization, we can generalize the so-called ***Slater determinant*** to

$$|\psi_A\rangle \quad = \quad \frac{1}{\sqrt{N!}}\begin{vmatrix} |u_1\,(1)\rangle & |u_2\,(1)\rangle & \cdots & |u_N\,(1)\rangle \\ |u_1\,(2)\rangle & |u_2\,(2)\rangle & \cdots & |u_N\,(2)\rangle \\ \vdots & \vdots & \vdots & \vdots \\ |u_1\,(N)\rangle & |u_2\,(N)\rangle & \cdots & |u_N\,(N)\rangle \end{vmatrix}. \tag{4.83}$$

This determinant will be found to vanish whenever $|u_j\rangle = |u_k\rangle$ for any $j \neq k$.

4.3.1 Interactions Between Direct and Exchange Processes

The symmetrization process in the analysis of identical particles can also bring coherence effects (or interferences), which are not present when the particles are different (or, at the least, are distinguishable). Let us consider the case of two identical particles (either bosons or fermions) with initial states $|\varphi\rangle$ and $|\chi\rangle$ (of course, we do not know which particle is in what state). The initial state of the overall system is then given by

$$|\psi_{\mathrm{i}}\rangle = \frac{1}{\sqrt{2}}\left(\hat{1} + \epsilon\hat{P}_{21}\right)|\varphi\,(1)\,\chi\,(2)\rangle\,, \tag{4.84}$$

where $\epsilon = \pm 1$ depending on the type of particles we are dealing with. The two particles, being identical, share the same basis $\{|u_k\,(j)\rangle\}$, which are eigenvectors for the extended observables $\hat{\mathbb{B}}\,(j)$ ($j = 1, 2$). We now inquire about the probability of simultaneously measuring the eigenvalues b_m and b_n on the system (i.e, any one of the particle in the state $|u_m\rangle$ and the other in the state $|u_n\rangle$). The eigenvector associated with these eigenvalues must also be symmetrize, since the particles are indistinguishable, and

$$|\psi_{\mathrm{f}}\rangle = \frac{1}{\sqrt{2}}\left(\hat{1} + \epsilon\hat{P}_{21}\right)|u_m\,(1)\,u_n\,(2)\rangle\,. \tag{4.85}$$

We then calculate

$$
\begin{aligned}
\langle \psi_{\mathrm{f}} \,|\, \psi_{\mathrm{i}} \rangle &= \frac{1}{2} \left\langle u_m(1)\, u_n(2) \middle| \left(\hat{1} + \epsilon \hat{P}_{21}^{\dagger}\right) \left(\hat{1} + \epsilon \hat{P}_{21}\right) \middle| \varphi(1)\, \chi(2) \right\rangle \\
&= \left\langle u_m(1)\, u_n(2) \middle| \left(\hat{1} + \epsilon \hat{P}_{21}\right) \middle| \varphi(1)\, \chi(2) \right\rangle \\
&= \langle u_m(1)\, u_n(2) \,|\, \varphi(1)\, \chi(2) \rangle + \epsilon \langle u_m(1)\, u_n(2) \,|\, \chi(1)\, \varphi(2) \rangle \\
&= \langle u_m \,|\, \varphi \rangle \langle u_n \,|\, \chi \rangle + \epsilon \langle u_m \,|\, \chi \rangle \langle u_n \,|\, \varphi \rangle . \qquad (4.86)
\end{aligned}
$$

We thus find that the projection of the two states on one another yields two terms commonly called ***direct process*** and ***exchange process*** term, as the initial states of the particles $|\varphi\rangle$ and $|\chi\rangle$ are swapped in their projections on the eigenvectors $|u_m\rangle$ and $|u_n\rangle$ between the two terms. The probability of finding the eigenvalues b_m and b_n is therefore

$$
\begin{aligned}
\mathcal{P}(b_m, b_n) &= \left| \langle u_m \,|\, \varphi \rangle \langle u_n \,|\, \chi \rangle + \epsilon \langle u_m \,|\, \chi \rangle \langle u_n \,|\, \varphi \rangle \right|^2 \\
&= \left| \langle u_m \,|\, \varphi \rangle \right|^2 \left| \langle u_n \,|\, \chi \rangle \right|^2 + \left| \langle u_m \,|\, \chi \rangle \right|^2 \left| \langle u_n \,|\, \varphi \rangle \right|^2 \\
&\quad + 2\epsilon \mathrm{Re} \left\{ \langle u_m \,|\, \varphi \rangle \langle u_n \,|\, \chi \rangle \langle u_m \,|\, \chi \rangle^* \langle u_n \,|\, \varphi \rangle^* \right\} . \qquad (4.87)
\end{aligned}
$$

The last term in equation (4.87) is clearly an interference term.

We can contrast this result with the case when the two particles are not identical (or can be distinguished), for which case we simply set $\epsilon = 0$ equation (4.87). The interference term then disappears.